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A NECESSARY CONDITION FOR WYE-DELTA TRANSFORMATION

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ABSTRACT

Let the binary operations $\cup$ and $\cap$ denote the arithmetic operations of addition $(a \cup b = a + b)$ and reciprocal addition $(a \cap b = ab/(a + b))$, that is, $\cup$ and $\cap$ are the series and parallel combination rules for resistor networks. It is easy to verify that the identity

$$(1) \qquad (a \cup b \cup c) \cap ((a \cap b) \cup (a \cap c) \cup (b \cap c)) = (a \cap b \cap c) \cup ((a \cup b) \cap (a \cup c) \cap (b \cup c))$$

holds. Furthermore if the binary operation $\cdot$ denotes arithmetic multiplication then the identity

$$(2) \qquad (a \cup b) \cdot (a \cap b) = a \cdot b$$

holds. Now suppose only that $\cup$ and $\cap$ are associative commutative binary operations on an abstract set S and that there exists a commutative group $R, \cdot$ such that S is a subset of R and (2) holds on S. If in an abstract network theory $\cup$ and $\cap$ are the series and parallel combination rules and if wye-delta transformations are valid in this theory then it is shown that (1) must hold. Wye-delta transformations are valid for resistor networks but need not exist in resistor-inductor-capacitor (RLC) networks. Thus, in the presence of (2), identity (1) is a necessary but not sufficient condition for wye-delta transformation.

A NECESSARY CONDITION FOR WYE-DELTA TRANSFORMATION

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1. Introduction

Let the binary operations $\cup$ and $\cap$ denote the arithmetic operations of addition $(a \cup b = a + b)$ and reciprocal addition $(a \cap b = ab/(a + b))$, that is $\cup$ and $\cap$ are the series and parallel combination rules for resistor networks. It is easy to verify that the identity

$$(1) \qquad (a \cup b \cup c) \cap ((a \cap b) \cup (a \cap c) \cup (a \cap c)) = (a \cap b \cap c) \cup ((a \cup b) \cap (a \cup c) \cap (b \cup c))$$

holds. Furthermore if the binary operation $\cdot$ denotes arithmetic multiplication then the identity

$$(2) \qquad (a \cup b) \cdot (a \cap b) = a \cdot b$$

holds. Now suppose only that $\cup$ and $\cap$ are associative commutative binary operations on an abstract set and that there exists a commutative group $R, \cdot$ such that S is a subset of R and (2) holds on S. If in an abstract network theory (defined later) $\cup$ and $\cap$ are the series and parallel combination rules and if wye-delta transformations are valid in this theory then it will be shown that (1) must hold. Wye-delta transformations are valid for resistor networks but need not exist in resistor-inductor-capacitor (RLC) networks. Thus, in

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the presence of (2), identity (1) is a necessary but not sufficient condition for wye-delta transformation.

As a second example, let $\cup$ and $\cap$ be given arithmetically by $a \cup b = ab$ and $a \cap b = a + b - ab$. These are the series and parallel rules for combining the connection probabilities of independent randomly-operating switches. Identity (2) holds where $\cdot$ is arithmetic addition but (1) fails for all a, b, c where $0 < a, b, c < 1$. Consequently wye-delta transformation is not valid for these probabilistic networks. Nevertheless certain approximate transformations can be of use. The details are given in [5].

Similar results obtain for the operations $a \cup b = a + b - ab$ and $a \cap b = ab/(a + b - ab)$, where $\cdot$ is multiplication, and for $a \cup b = \min(a + b, 1)$ and $a \cap b = \max(a + b - 1, 0)$, where $\cdot$ is addition. The first pair of operations are the series (cascade) and parallel (diversity reception [1]) rules for combining the noise powers of one-watt communication channels with incoherent additive noise. The second pair of operations occurs in multivalued logics [2, 7]. In particular the values 0, 1 and 1/2 correspond to the short, cut and neutral configurations of the Shannon switching game [4]. Both pairs of operations extend to bridge-type networks. In consequence of the failure of (1), wye-delta transformation is not valid for these networks.

The preceeding examples illustrate the use of identity (1) in determining the possibility of wye-delta transformation given only series and parallel combination rules. The only known examples having network interpretation and satisfying both (1) and (2) are algebraically similar to resistor networks in that the operation $\cdot$ distributes over both $\cup$ and $\cap$. Thus the identity (2) seems of limited applicability. Nevertheless, its derivation is of value in demonstrating the use of abstract algebra in network theory.

The terms "network theory" and "wye-delta transformation" are commonly associated with linear electrical devices. They will now be defined in an abstract fashion.

2. Definition of an abstract network theory

Let S be a fixed abstract set containing at least two elements. G will denote any finite connected linear graph. It is also assumed that G is undirected and contains at least two vertices. The vertices of G will be indexed by $1, \ldots, m$ and the branches by $1, \ldots, n$. This indexing need not be consecutive. For each branch i let a_i be a member of S. a_i will be called the branch value. A graph G together with its branch values $a_1, \ldots, a_n$ constitute a network.

For each network $G, a_1, \ldots, a_n$ and each pair of branch indices i and j where $1 \leqq i < j < m$, let $V_{ij}^G(a_1, \ldots, a_n)$ be a member of S. $V_{ij}^G(a_1, \ldots, a_n)$ is called the driving-point value between i and j. The driving-point values are also subject to the following three requirements.

(3) The driving-point values are to be the same for isomorphic networks, that is, they are to be independent of the particular indexing of the vertices and branches.

(4) The deletion of any branch which connects only one vertex (a loop) does not change any of the driving-point values.

(5) If a branch i connects vertices j and k and if the removal of that branch leaves the graph disconnected, then $V_{jk}^G(a_1, \ldots, a_n) = a_i$ must hold.

The set S, all graphs G, and the functions V_{ij}^{G} constitute a network theory.

In the case of resistor networks, S is the set of positive numbers, a_i is the branch resistance of the i^{th} branch, and V_{ij}^{G} is the driving-point resistance measured between terminals i and j. The presence of a branch connected to only one terminal does not affect any driving-point resistance (postulate 4). If the resistor i is the only connection between distinct terminals j and k then the driving-point resistance equals the branch resistance (postulate 5). The customary transfer resistances can be obtained from the driving-point resistances [3]. If, instead, S is the set of complex numbers with positive real part and if the term "impedance" is used in place of "resistance", then these statements also apply to resistor-inductor-capacitor networks.

A network theory is said to be reciprocal if there exist binary operations $\cup$, $\cap$, and $\cdot$ which satisfy the following four requirements.

(6) $\cup$ and $\cap$ are associative commutative binary operations defined on S ($\cup, \cap: S^2 \rightarrow S$).

(7) Suppose that a network contains two branches in series whose branch values are a and b. If this series combination is replaced by a single branch of value $a \cup b$ then the resulting driving-point values are the same as the corresponding values in the original network. (The number of vertices, however, is decreased by one.)

(8) Suppose that a network contains two branches in parallel whose branch values are a and b. If this parallel combination is replaced by a single branch of value $a \cap b$ then the resulting driving-point values are the same as the corresponding values in the original network. (The number of vertices remains the same.)

(9) There exists a commutative group $R, \cdot$ such that

S is a subset of R and

$(a \cup b) \cdot (a \cap b) = a \cdot b$ holds for all members

a and b of S.

The operations $\cup$ and $\cap$ are called the series and parallel combination rules of the network theory. They can be obtained, uniquely, as the driving-point values of two-branch series and parallel networks. This equivalence of series and parallel combinations with a single branch is a well-known property of resistor networks.

Only the existence of the reciprocity operation $\cdot$ is required. Yet this existence may be difficult to determine. Separated from its network terminology, the unsolved problem is this:

> Given a set S, closed under two associative, commutative operations $\cup$ and $\cap$, when does there exist a commutative group $R, \cdot$ such that S is contained in R and $(a \cup b) \cdot (a \cap b) = a \cdot b$ holds for all a and b in S?

In the case of resistor and resistor-inductor-capacitor networks, postulate (9) is satisfied by the multiplicative group of non-zero complex numbers. For these or any other reciprocal networks the reciprocity operation $\cdot$ cannot be unique. To demonstrate, let c be any fixed member of R other than the identity and define the operation $\otimes$ reciprocity operation $\cdot$ cannot be unique. To demonstrate, let c be any fixed member of R other than the identity and define the operation $\otimes$ by $a \otimes b = a \cdot b \cdot c$. Then $R, \otimes$ is also a commutative group and the identity $(a \cup b) \otimes (a \cap b) = a \otimes b$ holds. There are other possibilities. If, as in the theory of switching networks, the identity $a \cup a = a$ holds then $S, \cup, \cap$ is a distributive lattice. Suppose it is also the lattice of a commutative lattice-ordered group. Then this group operation is a suitable reciprocity operation. Any distributive lattice can be realized as a lattice of sets and the symmetric difference operation (set addition modulo 2) is a reciprocity operation. For the previous operation R equals S while for this last operation it is considerably larger.

Customarily the term "reciprocity" refers to the invariance of transmission under the interchange of source and sink. Here it is used in a weaker sense: Consider the network G of Figure 1 and the networks $G^{\#}$ and G^{*} which are obtained from G by coalescing vertices 1 and 2 and 1 and 3 respectively. Then, using postulates (3) through (9),

$$V_{12}^{G}(a, b) \cdot V_{13}^{G^{\#}}(a, b) = (a \cup b) \cdot (a \cap b) = a \cdot b = V_{12}^{G^{*}}(a, b) \cdot V_{13}^{G}(a, b)$$

and hence

$$V_{12}^{G}/V_{12}^{G^{*}} = V_{13}^{G}/V_{13}^{G^{\#}} \tag{10}$$

holds. The division bar indicates multiplication $(\cdot)$ by the group inverse. Stated more generally, the ratio (in $R, \cdot$) of the driving-point value between 1 and 2 and the driving-point value between 1 and 2 with 1 and 3 "short-circuited" is the same as the corresponding ratio obtained by interchanging terminals 2 and 3. From the assumptions already made, principally (6) and (9), this reciprocity relation holds for series-parallel networks, that is where the driving-point values can be expressed as polynomials in $\cup$ and $\cap$. For example, considering the graphs G, $G^{\#}$ and G^{*} of Figure 2,

$$V_{12}^{G}/V_{12}^{G^{*}} = a\cap(b\cup(c\cap(d\cup(e\cap(f\cup g)))))/a\cap(b\cup(c\cap(d\cup(e\cap f)))) =$$

$$g\cap(f\cup(e\cap(d\cup(c\cap(b\cup a)))))/g\cap(f\cup(e\cap(d\cup(c\cap b)))) = V_{13}^{G}/V_{13}^{G^{\#}} .$$

In the case of linear electrical networks the customary type of reciprocity is given by the equation $Z_{12} = Z_{21}$. It is not needed* for the weaker relation $Z_{11}Y_{11} = Z_{22}Y_{22}$ which is the type of reciprocity expressed by (10).

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*Since linear electrical networks consisting of two-terminal components are automatically reciprocal [8], this distinction becomes clear only when multi-terminal elements are considered. Networks of these elements are generally unsuitable for wye-delta transformation. An exception is the unistor (or gyristor) [6]. While unistor networks are non-reciprocal (in the usual electrical sense) and admit wye-delta transformations, they do not have a commutative series operation and hence identity (1) is not relevant.

The previous network definitions could have been expressed more formally by changing the graphical terminology to that of relations or of incidence matrices. Or the definitions could have been made more abstract by the use of matroids [9] instead of graphs. Either approach would obscure the geometric character of the wye-delta transformation.

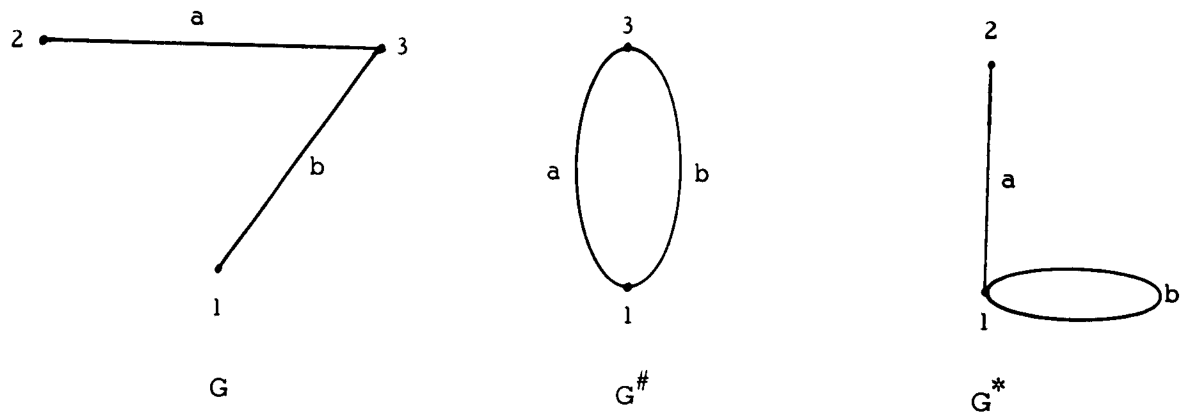

Figure 1

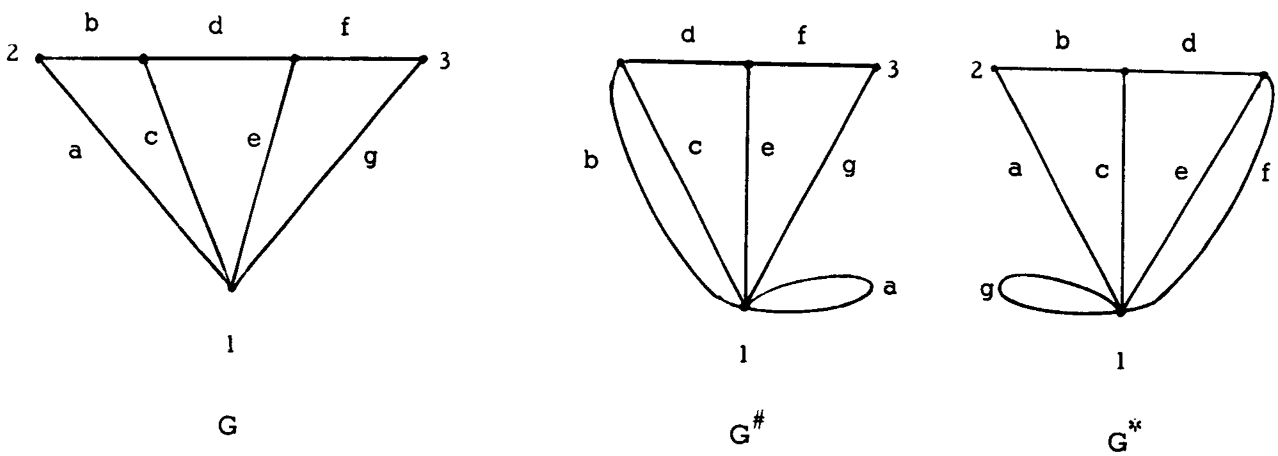

Figure 2

3. Definition of wye-delta transformation

A wye is a graph or subgraph consisting of three branches which connect each of three vertices to a common fourth vertex. This fourth vertex is not otherwise connected. A delta is a graph or subgraph consisting of three branches which connect each of three vertices so as to form a circuit. Wye and delta networks are illustrated in Figure 3.

Assume that a fixed correspondence is established between the three vertices of the wye, not including the common vertex, and those of the delta. A network theory admits valid wye-delta transformations if the following two properties hold.

(11) For each set a, b, c of wye branch values there exists at least one corresponding set a', b', c' of delta branch values. Moreover, each possible set of delta branch values corresponds to at least one set of wye branch values.*

(12) Suppose that a network contains a wye whose branch values are a, b, c. If this wye is replaced by a delta whose branch values are a', b', c' then the resulting driving-point values are the same as the corresponding values in the original network. This replacement must preserve the correspondence

*The validity of wye-to-delta transformations requires a set a', b', c' for each a, b, c. The validity of delta-to-wye transformations requires a set a, b, c for each a', b', c'. Thus the validity of both transformations yields a correspondence of equivalence classes of wye values a, b, c and delta values a', b', c' which is one-to-one and onto.

between the vertices of the wye and delta. Furthermore, the equivalence of the driving-point values of the two networks must be preserved when any two corresponding vertices of each network (not including the common wye vertex) are coalesced. These results must hold for each set a', b', c' which corresponds to a, b, c.

As a consequence of (12), the driving-point values between vertices 2 and 3 of the wye and delta networks of Figure 3 must be the same. By (7) and (8) this results in the equation $b' \cup c' = a \cap (b \cup c)$. Similarly, coalescing vertices 2 and 3 of each graph yields the networks of Figure 4 and equating the driving-point values between vertices 1 and 2 yields the equation $a' \cup (b' \cap c') = b \cap c$. Thus by symmetry,

$$(13) \qquad \begin{aligned} a \cup b &= c' \cap (a' \cup b') & \qquad a' \cap b' &= c' \cap (a' \cup b') \\ a \cup c &= b' \cap (a' \cup c') & \qquad a' \cap c' &= b' \cap (a' \cup c') \\ b \cup c &= a' \cap (b' \cup c') & \qquad b' \cap c' &= a' \cap (b' \cup c') \end{aligned}$$

all hold.

The series and parallel postulates (6), (7) and (8) are limiting cases of the wye-delta postulates (11) and (12). The series postulate results when one of the wye branches is deleted. The parallel postulate results when one of the delta branches is replaced by a short-circuit. For resistor networks, the wye-delta transformation correspondence is given by

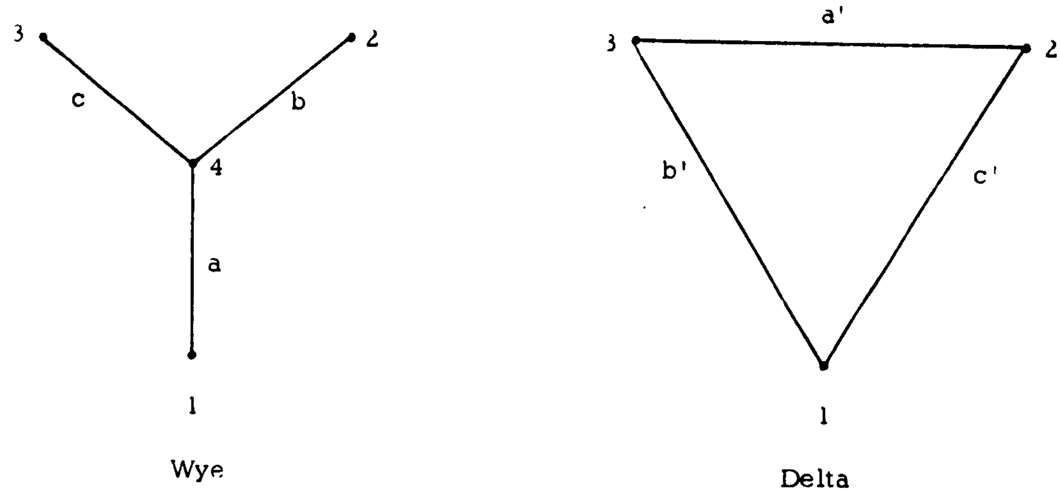

Figure 3

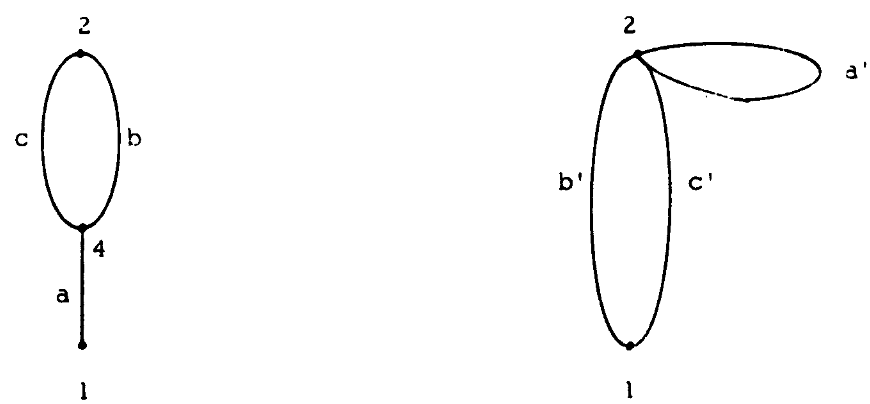

Figure 4

$a' = (ab + ac + bc)/a$, $b' = (ab + ac + bc)/b$ and $c' = (ab + ac + bc)/c$. Letting a become infinite yields the series operation $a' = b + c$ and letting a' approach zero yields the parallel operation $a = b'c'/(b' + c')$.

Wye-delta transformation fails for resistor-inductor-capacitor networks only in that the transformed values a', b', c' (and a, b, c) may have negative (or zero) real-part. For example, if $a = 1 + 2i$ and $b = c = 1 - 2i$ then $a' = -(1 + 18i)/5$. This failing is of electrical importance in that such transformations require the use of active (energy-producing) components. Any extension of S into the negative half of the complex plane yields a non-zero complex number z such that z, $-z$ and $2z$ are members of S. Using the notation of (10) and the network G of Figure 5, $V_{12}^{G} = \frac{4}{3}z$, $V_{13}^{G} = \frac{1}{3}z$ and $V_{12}^{G^*} = V_{13}^{G^\#} = 0$. This contradicts the cancellation property of (10) and hence no reciprocity operation $\cdot$ can exist. Furthermore, in the wye network G^o, exactly two of the three driving-point values between vertices 1, 2 and 3 are zero. This behavior requires infinite energy and violates Kirchoff's laws.

Now consider the network G_1 of Figure 6. By (7) and (8), that is, contracting series and parallel subgraphs, the driving-point value between vertices 1 and 6 is found to be $a \cup b \cup ((a \cup c) \cap (b \cup c))$. An application of the wye-delta transformation of Figure 3 eliminates vertex 4 and yields the network G_2. A second application eliminates vertex 5 and yields the network G_3. By (12) the driving-point value between vertices 1 and 6

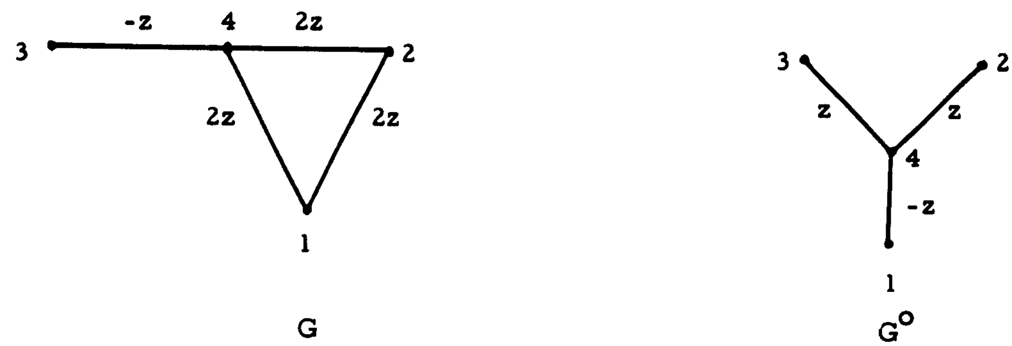

Figure 5

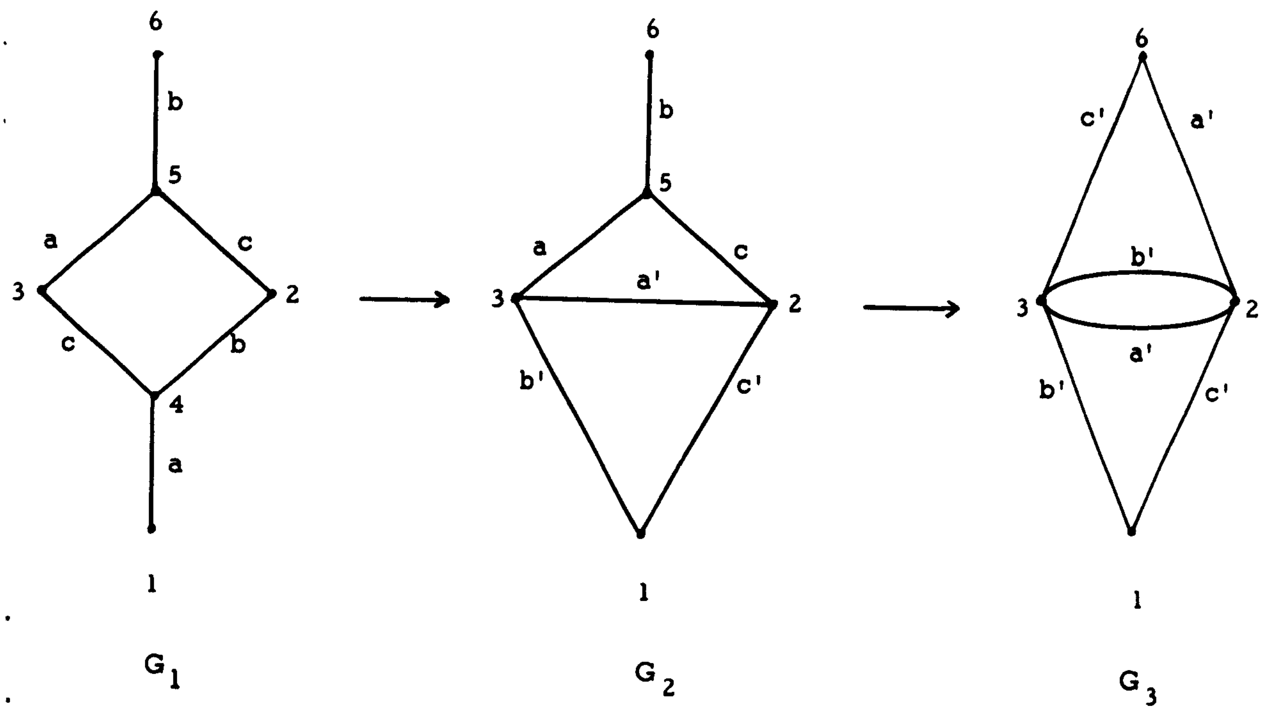

Figure 6

of G_3 is the same as that for G_1. This value can also be found from G_3 by means of (10), the reciprocity relation:

First, replace the parallel combination in G_3. This yields the network G_4 of Figure 7. Next, application of the wye-delta transformation of Figure 8, where d, e, f are delta values corresponding to wye values a', c', $a'\cap b'$, yields the network G_5. Finally, application of the parallel operation yields G_6. By (6) and (9),

$$f\cap((b'\cap c)\cup(c'\cap e))/(f\cap c'\cap e) =$$

$$(f\cup(c'\cap e))\cdot(b'\cap d)\cup(c'\cap e)/f\cup((b'\cap d)\cup(c'\cap e))\cdot(c'\cap e) =$$

$$b'\cap d\cap(f\cup(c'\cap e))/(b'\cap d\cap c'\cap e) .$$

$f\cap((b'\cap d)\cup(c'\cap e))$ and $b'\cap d\cap(f\cup(c'\cap e))$ are respectively the driving-point values between vertices 1 and 6 and 1 and 3 of G_6. The same values derived from G_1 and G_3 are $a\cup b\cup((a\cup c)\cap(b\cup c))$ and $b'\cap(c'\cup(a'\cap b'\cap(a'\cup c')))$. Similarly, $f\cap c'\cap e$ and $b'\cap d\cap c'\cap e$ are the corresponding driving-point values where vertices 1 and 3 or 1 and 6 are coalesced. In G_3 they equal, respectively, $c'\cap(a'\cup(c'\cap a'\cap b'))$ and $b'\cap c'\cap((a'\cap b')\cup(a'\cap c'))$. The result is the equation

$$(14)\qquad a\cup b\cup((a\cup c)\cap(b\cup c))/c'\cap(a'\cup(c'\cap a'\cap b')) =$$

$$b'\cap(c'\cup(a'\cap b'\cap(a'\cup c')))/b'\cap c'\cap((a'\cap b')\cup(a'\cap c')) .$$

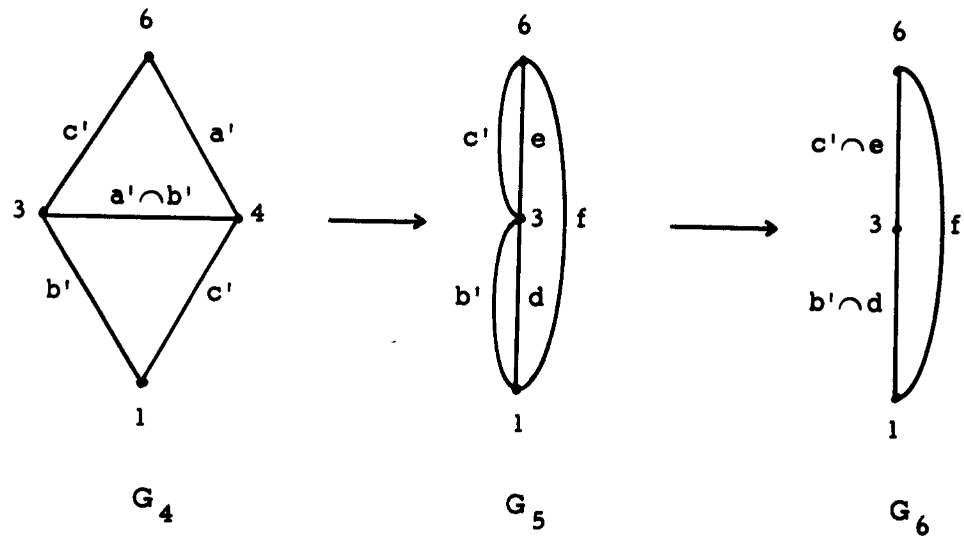

Figure 7

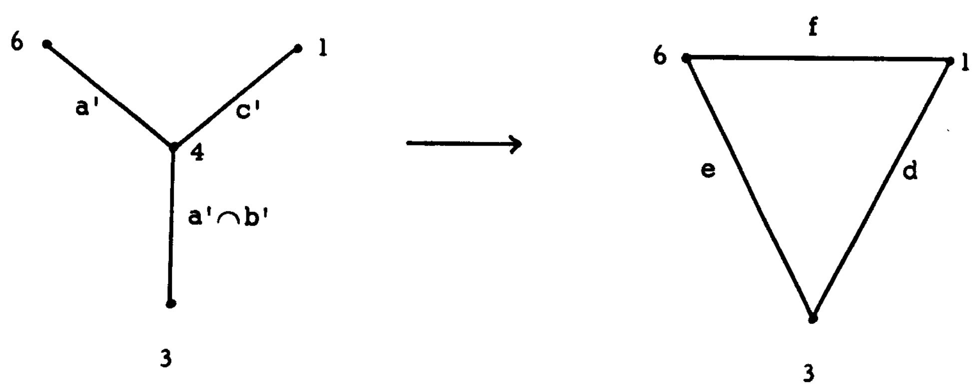

Figure 8

By several applications of identity (2), equation (14) can be transformed into

$$(15)\qquad a\cup b\cup((a\cup c)\cap(b\cup c))/(a'\cap b')\cup(a'\cap c')\cup(b'\cap c') =$$

$$a'\cap b'\cap(a'\cup c')\cap(b'\cup c')/(a'\cap b')\ .$$

The symbols a, b, c can be eliminated from (15) by use of the first column of equations in (13). Application of (2) then yields the symmetric equation

$$(16)\qquad a'\cdot b'\cdot c'/(a'\cap b')\cup(a'\cap c')\cup(b'\cap c') =$$

$$a'\cap b'\cap c'\cap(a'\cup b')\cap(a'\cup c')\cap(b'\cup c')\cdot(a'\cup b'\cup c')\ .$$

Similarly, using (13) to eliminate a', b', c' from (15) results in the dual equation

$$(17)\qquad a\cdot b\cdot c/(a\cup b)\cap(a\cup c)\cap(b\cup c) =$$

$$a\cup b\cup c\cup(a\cap b)\cup(a\cap c)\cup(b\cap c)\cdot(a\cap b\cap c)\ .$$

Since the primed values a', b', c' and the unprimed values a, b, c have the same range, equation (16) implies

$$(18)\qquad a\cdot b\cdot c/(a\cap b)\cup(a\cap c)\cup(b\cap c) =$$

$$a\cap b\cap c\cap(a\cup b)\cap(a\cup c)\cap(b\cup c)\cdot(a\cup b\cup c)\ .$$

Eliminating $a \cdot b \cdot c$ between equations (17) and (18) yields

$$a \cup b \cup c \cup (a \cap b) \cup (a \cap c) \cup (b \cap c) \cdot (a \cap b \cap c) \cdot (a \cup b) \cap (a \cup c) \cap (b \cup c) =$$

$$a \cap b \cap c \cap (a \cup b) \cap (a \cup c) \cap (b \cup c) \cdot (a \cup b \cup c) \cdot (a \cap b) \cup (a \cap c) \cup (b \cap c)$$

and by identity (2) it follows that

(1) $\quad (a \cup b \cup c) \cap ((a \cap b) \cup (a \cap c) \cup (b \cap c)) = (a \cap b \cap c) \cup ((a \cup b) \cap (a \cap c) \cap (b \cup c)).$

4. Discussion

The previous section concluded with the proof of the following result.

Suppose that the value set S *and the series and parallel operations* $\cup$ *and* $\cap$ *are associated with a reciprocal network theory which admits valid wye-delta transformations. Then the balance condition*

$$(1)\qquad (a\cup b\cup c)\cap((a\cap b)\cup(a\cap c)\cup(b\cap c)) = (a\cap b\cap c)\cup((a\cup b)\cap(a\cup c)\cap(b\cup c))$$

holds for all members a, b, c *of* S.

The identity (1) is equivalent to the assertion that the driving-point values between vertices 1 and 2 of the two networks of Figure 9 are both equal. These two networks can be obtained from the network of Figure 10 by coalescing vertices 3 and 4 or 5 and 6 respectively. Thus the driving-point value bet ween vertices 1 and 2 of the network of Figure 10 is the same if either vertices 3 and 4 or vertices 5 and 6 are "short-circuited". For this reason, (1) is called a *balance* condition.*

Many network theories which fail to satisfy the balance condition, fail to satisfy it in the special case where $a = b = c$. This case is easily verified. When any of the values a, b, c (or a', b', c') correspond to an

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*In the case of resistor networks, if a potential is applied between vertices 1 and 2 of the network of Figure 10 then vertices 3 and 4 have equal potential if and only if $a^2 = bc$. This same condition holds for vertices 5 and 6. Thus, for either vertex-pair, the customary "balanced bridge" occurs only if $a^2 = bc$.

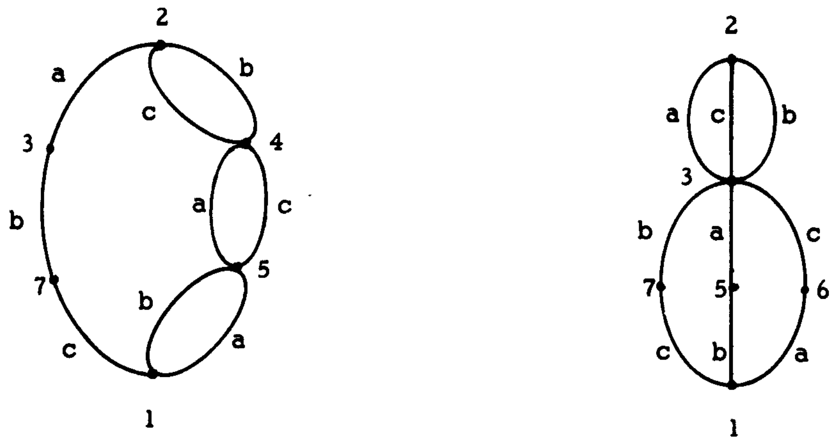

Figure 9

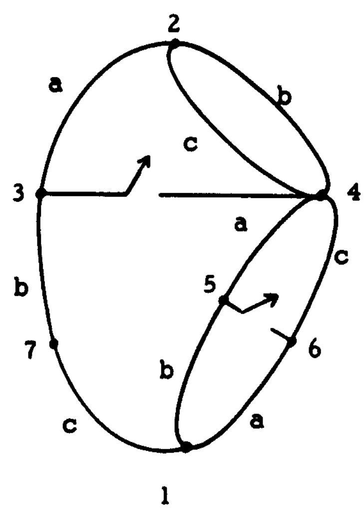

Figure 10

open or short-circuit then the wye-delta transformation is merely series-parallel replacement. Hence for such values a, b, c the balance condition is of no constraint.

A pair of associative commutative operations can satisfy the balance condition and not be associated with any reciprocal network theory. For example, let the operations $\cup$ and $\cap$ be given by $a \cup b = a + b$ and $a \cap b = ab$ or by $a \cup b = a \cap b = a + b$. Using the first pair of operations, (10) fails for the network G of Figure 5 when all four branch values are changed to 1. Using the second pair of operations, (10) fails for the network G^{o} of Figure 5 whenever all three branch values are made unequal. Moreover, it is not possible to satisfy all six equations in (13). Hence any network theory having either pair of operations as its series and parallel combination rules can neither be reciprocal nor admit valid wye-delta transformations.

Another example is that of minimal cost networks. S is the set of non-negative numbers and $\cup$ and $\cap$ are given by $a \cup b = a + b$ and $a \cap b = \min(a, b)$. This network theory satisfies the balance condition but cannot be reciprocal as it violates the cancellation property: $a \cup b = a$ if and only if $a \cap b = b$. It does admit valid wye-delta transformations. However, the correspondence between the wye values a, b, c and the delta values a', b', c' is one-to-one only for <u>positive</u> numbers a, b, c. By duality, similar statements hold for capacity networks.

The reciprocity operation · was used extensively in the proof of the balance condition and there is no indication that postulate (9) is unnecessary. Hence it is conjectured that there exists a network theory which admits valid wye-delta transformations, satisfies postulates (6), (7) and (8), but fails to satisfy either (9) or the balance condition. Few of the known reciprocal network theories satisfy the balance condition. Any example whose series and parallel operations are essentially different from those of resistor networks would be of interest.

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